Rotations of Space and Coordinate Transformations

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Abstract: Dealing with orientation or geometrical restitution of pictures in photogrammetry also means dealing with rotations and transformations. Therefore the assumption that all the respective questions should have been solved properly already seems to be logical. But in practice considerable problems arise when discrete values have to be transformed for real applications, especially when measured angles, e. g. of an inertial navigation system or of a photo theodolite have to be included in the calculations. Even photogrammetric textbooks and publications cannot grant support in every single case, as not all of them show the basic mathematical relations correctly.

For this reason the aim of this article is to promote the comprehension for rotations of space, coordinate transformations and measured angle values. It describes the basic relations and gives suggestions for practical applications and parameterisation of rotations. The indicated formulas grant a successful application especially for e. g. inertial navigation systems like AEROcontrol as well as for terrestrially measured orientation angles.

Zusammenfassung: Drehungen und Transformationen begegnen uns in der Photogrammetrie ständig, wenn es um die Orientierung oder geometrische Auswertung von Bildern geht. Man sollte also meinen, dass damit verbundene Fragestellungen vollständig beherrscht werden. In der Praxis zeigt sich jedoch immer wieder, dass erhebliche Probleme auftreten, sobald diskrete Werte für reale Anwendungen umzusetzen sind. Dies gilt insbesondere, wenn gemessene Winkel z.B. eines Inertialsystems oder eines Phototheodoliten in Berechnungen einbezogen werden sollen. Auch photogrammetrische Lehrbücher und Veröffentlichungen sind hier nicht immer eine Hilfe, da nicht alle die grundlegenden mathematischen Beziehungen korrekt darstellen.

Dieser Artikel soll das Verständnis für räumliche Drehungen, Koordinatentransformationen und gemessene Winkelwerte fördern. Er beschreibt daher die grundsätzlichen Beziehungen und gibt Empfehlungen zur praktischen Anwendung und zur Parametrisierung der Drehungen. Die angegebenen Formeln garantieren einen erfolgreichen Einsatz insbesondere z.B. für Inertialsysteme wie AEROcontrol sowie auch für terrestrisch gemessene Orientierungswinkel.

1 Introduction

As a student of geodesy I never managed to develop a real deep understanding for the questions of rotated and fixed axes in terms of photogrammetry. As a newly graduated assistant at the Hannover University Institute for Photogrammetry and Engineering Surveys I was assigned to work in the department for Engineering Surveys and Close Range Photogrammetry. In the frame of preliminary works for the renovation of old parts of the town of Hameln, one of my first tasks was to integrate measured orientation angles of photogrammetric exposures into a bundle adjustment with the aim of reducing control points at the beautiful historical buildings. At that time the photos were mostly taken with an SMK 120, so that a consideration of the fixed relations between each photo pair seemed to be useful.

The necessary mathematical relations including the Taylor-linearizations were presented extensively in a thesis. I was able to develop a corresponding programme quite rapidly, as I already had collected experiences in software development for nine years. But unfortunately inexplicable tensions showed up in the results again and again. Therefore I doubted the correctness of the used formula [2]. Studying the photogrammetric textbooks wasn't of any help, too, because considerable contradictions arose.

Through contacts with one of the mathematical Institutes of the University I became aware of the book "Matrix" of R. Zurmühl [3]. Chapter 5 of this book clarified several things. Together with the "Handbook of Survey Science", volume III a/1 [1] I was able to formulate the mathematical equations correctly and to convert it into a successful software.

Over the years I met colleagues occasionally again and again, who had the same problems with understanding the mathematical equations. Partly this can be put down to the inappropriate choice of the terms "rotated axes" and "fixed axes". The knowledge of correct formulations seems to have been submerged over the years –

in book [1] you can find a correct presentation, but (for me) the reasons for this can only be understood correctly in relation with book [3].

When writing software for bundle adjustment, which contains only angles of the photogrammetric aerial photos, it does not matter how the formulations of the angles are carried out. If rotated or fixed axes, if rotation sequence φ, ω, κ or κ, φ, ω – doesn't play any role. The slope angles φ and ω are very small in usual aerial photos and a control of the values isn't possible anyway. The rotation matrix **R** in its usual collinearity equation (1) is furthermore independent from its parameterisation. I.e.: for a concrete measuring image there is exactly one single rotation matrix **R** of an image vector |x', y', c| which complies with the equation. It can be described unambiguously by a set of three rotation angles or Rodrigues-parameters.

$$\mathbf{X}-\mathbf{X_0} = \lambda\, \mathbf{R}\, \mathbf{x'} \qquad (1)$$

But to establish the equation between the angles of the rotation matrix **R** and measured orientations, an exact sequence of the rotations to form the rotation matrix is absolutely necessary to obtain a correct result. This is the case when using inertial angle measurement systems like AEROcontrol by IGI or the Applanix systems. Even in case of successive rotations, like they are usual in analytical and digital photogrammetric systems, a correct sequence has to be observed (e. g. rotations of a model into the ground system as well as from the model into the two images).

Due to the current importance of these questions for the calibration of INS-systems, I have decided to make a fundamental statement on this important topic.

2 Rotations of Space and Coordinate Transformations

Basically two different views have to be differentiated: the rotation of space and the transformation of coordinates [3].

Talking about rotations of space means to look at a point cloud in a given coordinate system. Without changing the location of this point cloud, we turn (and if necessary shift) the axes cross of the coordinate system and bring it into a new position relative to the point cloud (Fig. 1). This corresponds to the rotation around rotated axes. In my opinion however the term of rotations of space is much more accurate and applicable. According to [3] the rotation matrix is then located at the **new** vector.

$$\mathbf{a} = \mathbf{R}\, \mathbf{a}^R \qquad (2)$$

(2) therefore represents with $\mathbf{a}^R$ the vector **a** after the rotation of space (by rotated axes) by **R**.

Talking about the transformation of coordinates means to look at only one single coordinate system, whose axes are fixed in a space (fixed axes). The point cloud is being shifted within this space and changed in its orientation and position (Fig. 2). The mathematical description is:

$$\mathbf{a}^R = \mathbf{R}\, \mathbf{a} \qquad (3)$$

In (3) $\mathbf{a}^R$ is the - with **R** transformed - vector **a**. Both ideas are therefore inverse to each other.

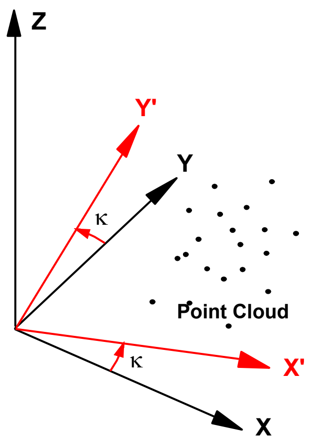

Fig. 1: Rotation of space by κ

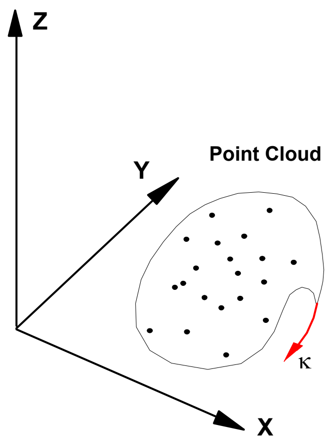

Fig. 2: Transformation of a point cloud

When executing several rotations in succession, (2) has to be applied repeatedly. Let $\mathbf{x}$ be a vector in the initial ground system (Ground, G). The space is being rotated by κ. This rotation of the initial system by κ results with $\mathbf{x}^{\kappa}$ in the vector $\mathbf{x}$ which is now presented in the rotated space. The next rotation of $\mathbf{x}^{\kappa}$ is carried out by φ (5). The matrix $\mathbf{R}_{\kappa}{}^{\varphi}$ goes on rotating the already by κ rotated space into another additionally by φ rotated space. The vector $\mathbf{x}$ now becomes $\mathbf{x}^{\kappa\varphi}$. The next rotation is effected with $\mathbf{R}_{\varphi}{}^{\omega}$ by ω out of the already by κ and φ rotated space. The vector $\mathbf{x}^{\kappa\varphi\omega}$ is now shown in a space which had been rotated by all the three angles (6).

$$\mathbf{x} = \mathbf{R}_{\mathbf{G}}{}^{\kappa}\, \mathbf{x}^{\kappa} \tag{4}$$

$$\mathbf{x}^{\kappa} = \mathbf{R}_{\kappa}{}^{\varphi}\, \mathbf{x}^{\kappa\varphi} \tag{5}$$

$$\mathbf{x}^{\kappa\varphi} = \mathbf{R}_{\varphi}{}^{\omega}\, \mathbf{x}^{\kappa\varphi\omega} \tag{6}$$

From putting (6) into (5) and the result into (4) follows (7):

$$\mathbf{x} = \mathbf{R}_{\mathbf{G}}{}^{\kappa}\, \mathbf{R}_{\kappa}{}^{\varphi}\, \mathbf{R}_{\varphi}{}^{\omega}\, \mathbf{x}^{\kappa\varphi\omega}$$

or shortly

$$\mathbf{x} = \mathbf{R}_{\kappa}\, \mathbf{R}_{\varphi}\, \mathbf{R}_{\omega}\, \mathbf{x}' \tag{7}$$

Vector $\mathbf{x}'$ in equation (7) therefore represents vector $\mathbf{x}$ in a space which had been rotated by κ, φ und ω.

It is obvious, that successive rotations of space can basically be effected by multiplying the rotation matrix, not by adding the angles. (7) can also be considered to be a continued transformation of $\mathbf{x}'$ to $\mathbf{x}$, but in this case with the inverse rotation sequence ω, φ, κ.

3 Parameterisation, singularities and rotation sequence

Equation (7) is always free of singularities, because all $\mathbf{R}$ are orthogonal matrix, i. e. their inverse is equal to the transponed and therefore exists in every case.

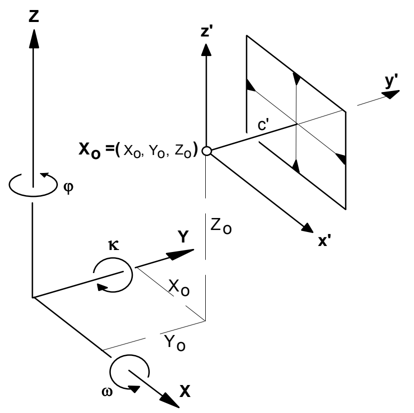

Fig. 3: Terrestrial rotation angles

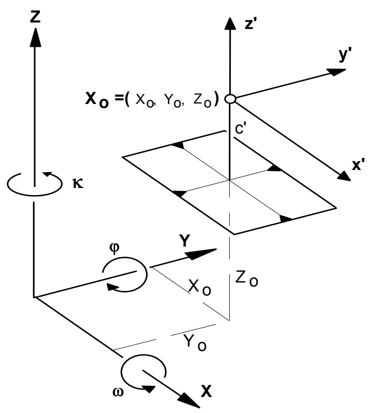

Fig. 4: Aerial photo rotation angle

But the *parameterisation* of $\mathbf{R}$ with rotation angles leads to problems regularly, when the second rotation angle becomes a rectangle. The first rotation axis is then turned exactly into the third rotation axis, so that the first and the third angle aren't independent from each other anymore. The consequence is that in this case not all space directions can be described with only one set of rotation angles. A remedy is offered e. g. by a parameterisation according to Rodrigues. This is recommended when an imagination of the parameters is unnecessary. Alternatively two different angle definitions can be chosen for the parameterisation, which offer the advantage of a very good graphicness:

Fig. 3. and Fig. 4. show a recommendation for the definition of rotation angles for terrestrial and aerial photos. In both definitions the rotation angles are zero for standard photographing cases. The choice of the rotation sequence is additionally crucial:

For terrestrial rotation angles, the rotation sequence φ, ω, κ is highly recommended for the establishment of a connection between the rotation angles of a photo and the directions measured by a photo theodolite for this photo. This improves the graphicness, too. Then φ is the horizontal angle, ω the vertical angle and κ the photo rotation. Then the collinearity equation for terrestrial photos is:

$$\mathbf{X}\text{-}\mathbf{X_0} = \boldsymbol{\lambda}\, \mathbf{R}_{\varphi}\, \mathbf{R}_{\omega}\, \mathbf{R}_{\kappa}\, \mathbf{x}' \tag{8}$$

For aerial photos the rotation sequences φ, ω, κ or ω, φ, κ are preferred. In the bundle programme BINGO, which I have developed, φ, ω, κ is being used. Today I would take this restriction differently, namely κ, φ, ω,, because then a good graphicness of the real photographing direction is granted also for sloping aerial photos.

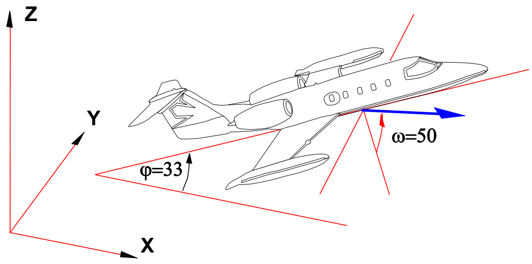

Fig. 5 Flight situation with κ = 0

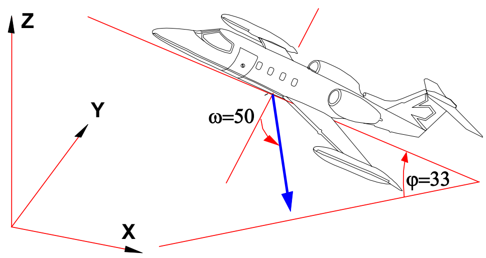

Fig. 6 Flight situation with κ = 250

Fig. 5 shows a photo airplane which takes good photos in climb with φ= 33. We assume oblique photos laterally out of the airplane with ω= 50 gon. When the flight direction is west-east, then κ= 0 gon. In this case also in the rotation sequence φ, ω, κ the angles are φ= 33 and ω= 50.

When changing the flight direction to κ= 250 (Fig. 6), the angles remain φ= 33 and ω= 50 only in case of the rotation sequence being κ, φ, ω. In case of the rotation sequence being φ, ω, κ, the angles are φ= -56, ω= -16 and κ= -156. Then the graphicness is lost completely. The reason for this is, that the κ-rotations are effected in a space, which had already been turned by φ and ω and therefore is lying obliquely in the space itself.

$$\mathbf{X}-\mathbf{X_0} = \lambda\, \mathbf{R}_\kappa\, \mathbf{R}_\varphi\, \mathbf{R}_\omega\, \mathbf{x}' \qquad (9)$$

For this reason the collinearity equation (9) is recommended for aerial photos.

4 Inertial systems

Even when calibrating inertial systems like AEROcontrol, the coordinate systems have to be chosen carefully. A special emphasis has to be put on the sequence of the rotation matrix. A not strict formulation of the equations easily leads to wrong results. In practice it can often be observed that even the rotation angles determined by an aerial triangulation are directly compared with the measured values of the IMU (inertial measurement unit) by subtraction. This is improper in any case. The rotation between IMU and camera can only be formulated by a rotation matrix.

Since always a little inclination remains between the axes of the IMU and of the camera, even when mounting the IMU on the camera very carefully, this inclination has to be determined by calibration. For the practice it is recommended to determine the reciprocal inclination of the axes directly in one step within the bundle triangulation. For this not even control points are necessary, as the GPS data support the photo unit sufficiently when they are sensibly arranged.

The IMU delivers per photo the orientation of the IMU – not of the photo.

Let the rotation matrix of the IMU $\mathbf{R_G}^\mathbf{I}$ for a photo be given, which turns the object system into the IMU-space. Every point $\mathbf{x}$ in the object space becomes $\mathbf{a}^\mathbf{I}$ in the IMU-space according to (10). The rotation of the object space into the photo is being described by (11).

$$\mathbf{x} = \mathbf{R_G}^\mathbf{I}\, \mathbf{a}^\mathbf{I} \qquad (10)$$

$$\mathbf{x} = \mathbf{R_G}^\mathbf{F}\, \mathbf{x}' \qquad (11)$$

The continued rotation of the photo space into the system of the IMU is being effected by (12). By putting (12) into (11) and equal it with (10) we get the equation (13) between photo and IMU. The matrix $\mathbf{R_F}^\mathbf{I}$ is the calibrated continued rotation of the photo into the system of the IMU.

$$\mathbf{x}' = \mathbf{R_F}^\mathbf{I}\, \mathbf{a}^\mathbf{I} \qquad (12)$$

$$\mathbf{R_G}^\mathbf{I} = \mathbf{R_G}^\mathbf{F}\, \mathbf{R_F}^\mathbf{I} \qquad (13)$$

This fundamental equation (13) is the basis for the calibration of the IMU. No matter if the elements of the matrix $\mathbf{R_F}^\mathbf{I}$ are determined at a test block or if a bundle block adjustment is to be stabilised by IMU-values, it is recommended for all cases to determine these matrix elements in the bundle block adjustment simultaneously. For this a liberalisation according to Taylor has to be effected, which leads to a quite extensive equation system. Φ, Ω, K are the measured angles of the IMU, $v\Phi$, $v\Omega$, vK its residuals, and $d\varphi$, $d\omega$, $d\kappa$ the searched calibration angles.

$$\begin{aligned} \Phi + v\Phi &= f_1\,(\varphi, \omega, \kappa, d\varphi, d\omega, d\kappa) \\ \Omega + v\Omega &= f_2\,(\varphi, \omega, \kappa, d\varphi, d\omega, d\kappa) \\ K + vK &= f_3\,(\varphi, \omega, \kappa, d\varphi, d\omega, d\kappa) \end{aligned} \qquad (14)$$

In the bundle block programme BINGO all these calibration angles are automatically considered as unknowns, as soon as IMU-angles are given as observations.

5 Choice of the coordinate system

An IMU delivers angles for each point immediately directly in the gravitation field of the earth. Every measured value therefore has its relation in a local tangential system, whose origin is in the respective measure point. The north direction is Geographically North, despite of very small rest errors. If the control point coordinates are available in UTM or in the Gauss-Krueger-system, the IMU-angles only have to be corrected by the meridian convergence to then transfer them strictly into such a system, too.

The measured values of the IMU do not give reasons to effect a transfer of all coordinates and angles into a local tangential system. This procedure partly suggested by other authors is without foundation respective the IMU-angles. At most the distorted coordinates of control points and DGPS-projection centres can give reason for such a procedure. This means, that persons who have adjusted the bundle blocks in these systems until today, despite of projection distortions of UTM or Gauss-Krueger, don't have to change their procedure.

Previous experiences had never justified in any case of a normal bundle adjustment a transition into a local tangential system.

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